Abstract: It is shown, that the crop-yield $z$ in dependence on two fertilizers $x$ and $y$ is the product of two components: $z$ in dependence on $x$ alone and $z$ in dependence on $y$ alone, divided by $c$, the yield without extern fertilizers, i.e. with $x=y=0$. For $n$ fertilizers, we have the product of $n$ components, divided by $c^{n-1}$.

Introduction

If only one fertilizer $x$ is used, the dependence of yield $z(x)$ on $x$ first was given by Mitscherlich (1909) in form of the differential equation

$$\frac{d\hat{z}}{dx} = b(a - \hat{z})$$

(1)

where $a$ is the asymptotic value of $z$, $b$ the factor of proportionality. As usual in statistics, $\hat{z}$ is the hypothetical value, $z$ the experimental value of the crop-yield. For equation (1) it is assumed: No over-fertilization. For the case of overfertilization see Schneeberger (2009b).

Solution of formula (1) with boundary condition $\hat{z}(x = 0) = c$ is Mitscherlich’s curve in the especial instructive form

$$\hat{z}(x) = c + (a - c)(1 - e^{-bx})$$

(2a)

demonstrated in figure 1 with $c=0.5$, $a=1.2$, $b=0.75$. For estimation of the parameters in practice see Schneeberger (2009a).

Figure 1: Crop-yield $\hat{z}(x)$ as function of one fertilizer $x$
An other form of formula (2a) is: 
\[ \dot{z}(x) = a - (a - c)e^{-bx} \] 
used in the following.

Baule (1918) gave the solution of equation (1) in the form

\[ \dot{z} = a(1 - e^{-b(x-d)}) \]  
with

\[ d = -\frac{1}{b}\ln\frac{a}{a-c} (<0) \]

in figure 1 we have \( d = -0.719 \).

**Generalization**

Now we assume two variables (=fertilizers) \( x \) and \( y \) (see figure 2)

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**Figure 2**: Crop-yield \( \dot{z}(x,y) \) as function of two fertilizers \( x \) and \( y \)
Then we have for the crop-yield $\hat{z}(x, y)$ with formula (2b)

$$\hat{z}(0, y) = c_1(y) = a_2 - (a_2 - c)e^{-b_2y} \quad (4a)$$
and

$$\hat{z}(x, 0) = c_2(x) = a_1 - (a_1 - c)e^{-b_2x} \quad (4b)$$

Note: For short we write $a_1(y = 0) = a_1$, $a_2(x = 0) = a_2$, $c_1(y = 0) = c_2(x = 0) = c$,

$b_1(y = 0) = b_1$, $b_2(x = 0) = b_2$,

$d_1(y = 0) = d_1$, $d_2(x = 0) = d_2$.

Generalizing formulae (4) we have

$$\hat{z}(x, y) = a_2(x) - (a_2(x) - c_2(x))e^{-b_2(x)y} \quad (5a)$$
and

$$\hat{z}(x, y) = a_1(y) - (a_1(y) - c_1(y))e^{-b_2(y)x} \quad (5b)$$

and herewith

$$\frac{a_2(x)}{a_1(y)} = \frac{1 - \frac{a_1(y) - c_1(y)}{a_1(y)}}{1 - \frac{a_2(x) - c_2(x)}{a_2(x)}} e^{-b_2(x)y} \quad (6)$$

Now I make use of a result of Mitscherlich (1947): “given different fertilizers, the parameter c (in Mitscherlich’s notation) is constant for a fixed fertilizer, as I could prove in tens of years of work”….. Mitscherlich’s “Wirkungsgrad” c is our parameter b except for the constant ln10. This means:

$$b_1(y) = b_1 \quad \text{and} \quad b_2(x) = b_2 \quad \text{independent of} \quad y \text{ resp.} \ x \quad (7)$$

Herewith we get from formula (6):

$$\frac{a_1(y) - c_1(y)}{a_1(y)} = 1 - \frac{c_1(y)}{a_1(y)} \quad \text{is independent of} \quad y, \ i.e. \quad \frac{a_1(y)}{c_1(y)} = \frac{a_1(y = 0)}{c_1(y = 0)} = \frac{a_1}{c} \quad (8a)$$

and

$$\frac{a_2(x)}{c_2(x)} = \frac{a_2}{c} \quad \text{independent of} \quad x \quad (8b)$$

and with this

$$d_1(y) = \frac{1}{b_1} \ln \frac{a_1(y)}{a_1(y) - c_1(y)} = \frac{1}{b_1} \ln \frac{a_1}{a_1 - c} = d_1 \quad \text{independent of} \quad y \quad (9a)$$
and

$$d_2(x) = d_2 \quad \text{independent of} \quad x \quad (9b)$$

Finally we have from formula (5a) (or (5b)):

$$\hat{z}(x, y) = \frac{a_2}{c} c_2(x) - \frac{a_2 - c}{c} c_2(x)e^{-b_2y} = \frac{1}{c} c_2(x)(a_2 - (a_2 - c)e^{-b_2y}) = \frac{1}{c} c_2(x)c_1(y) \quad (10)$$

or in the most instructive form

$$\hat{z}(x, y) = \frac{1}{c} (c + (a_1 - c)(1 - e^{-b_2x}))(c + (a_2 - c)(1 - e^{-b_2y})) \quad (11)$$
**RESULT:** The generalized Mitscherlich formula in two variables is the product of the one-dimensional formulae, multiplied by \(1/c\).

With formulae (9) one can show, that formula (11) is identical with the formula of Baule (1918).

\[
\hat{z}(x, y) = A(l - e^{-b_1(x - d_1)})(l - e^{-b_2(y - d_2)}) \quad \text{with} \quad A = \frac{a_1a_2}{c}
\] (12)

Equations (10) and (11) can easily be generalized for \(n\) fertilizers:

\[
\hat{z}(x_1, \ldots x_n) = \frac{1}{c^{n-1}}\hat{z}(x_1, 0, \ldots 0)\hat{z}(0, x_2, \ldots 0)\ldots\hat{z}(0, 0, \ldots x_n)
\] (13)

**Application**

The following data are from an example of Steinhauser, Langbehn and Peters (1992) with \(x\) (in 100 kg/ha of \(\text{P}_2\text{O}_5\)), \(y\) (in 100 kg/ha of \(\text{K}_2\text{O}\)), \(z\) (in 1000 kg/ha of rye).

<table>
<thead>
<tr>
<th>(y_1 = 0.25)</th>
<th>(y_2 = 0.50)</th>
<th>(y_3 = 0.75)</th>
<th>(y_4 = 1.00)</th>
<th>(y_5 = 1.25)</th>
<th>(y_6 = 1.50)</th>
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<td>(x_1 = 0.25)</td>
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<td>1.41</td>
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<td>(x_6 = 1.50)</td>
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<td>3.05</td>
<td>3.55</td>
<td>3.93</td>
<td>4.24</td>
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</tbody>
</table>

The 5 parameters \(c, a_1, b_1, a_2, b_2\) were determined with the method of Least Squares of Gauss

\[
f(c, a_1, b_1, a_2, b_2) = \sum_x \sum_y (z(x, y) - \hat{z}(x, y))^2 \rightarrow \text{Min}
\] (14)

summing over all 36 data-points. The minimum was gained iteratively with the non-linear Simplex-Method of Nelder and Mead (1965). The result is

\[
c = 0.2715, \quad a_1 = 0.9472, \quad b_1 = 0.899, \quad a_2 = 1.9438, \quad b_2 = 1.027, \quad \text{and}
\]

\[
\hat{z} = 3.6832(0.2715 + 0.6757(1 - e^{-0.899y}))(0.2715 + 1.6723(1 - e^{-1.027y}))
\]

In figure 3a the contour-lines \(\hat{z}(x, y) = 0.2, 0.4, \ldots 4.0\) are drawn, in figure 3b intersecting curves \(\hat{z}(x, y = \text{const.})\) for \(y = d_2, 0, 0.5, 1.0, 1.5\) and \(\infty\) are plotted. It is obvious that they are Mitscherlich-curves. We have \(d_1 = -0.376, \quad d_2 = -0.146\). The asymptotes of the curves \(\hat{z}(x, y = c)\) are horizontal dotted straight lines in figure 3b. Especially for \(y = \infty\) we get
\( \dot{z}(\infty, \infty) = \frac{a_1 a_2}{c} = 6.78 \), Baule’s parameter \( A \) in formula (12). \( \dot{z}(x, y = \infty) \) is curve \( a_2(x) \) of figure 2.
In analogy intersecting curves for \( x = \text{const} \) could be plotted.

**Figure 3a**: Contour-lines \( \dot{z}(x, y) = \text{const} \) - in 1000 kg/ha of rye, \( x \) in 100 kg/ha of \( P_2O_5 \), \( y \) in 100 kg/ha of \( K_2O \).
**Example:** With fertilizer 100 kg/ha of $\text{P}_2\text{O}_5$ (x=1) and 50 kg/ha of $\text{K}_2\text{O}$ (y=0.5) we get the crop-yield 2335 kg/ha of rye ($\hat{z} = 2.335$). With $x=y=0$ (i.e. without external fertilizers) we would get 272 kg/ha of rye ($\hat{z} = c = 0.272$).

**Acknowledgement**

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A generalization with overfertilization is given in paper 5 (Paper 5: Mitscherlich's Law: Generalization with several Fertilizers and Overfertilization)

**References**


