Mitscherlich's Law: Generalization with several Fertilizers

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Abstract: It is shown, that the crop-yield z in dependence on two fertilizers x and y is the product of two components: z in dependence on x alone and z in dependence on y alone, divided by c, the yield without extern fertilizers, i.e. with x=y=0. For n fertilizers, we have the product of n components, divided by c^{n-1}.

Introduction

If only one fertilizer x is used, the dependence of yield z(x) on x first was given by Mitscherlich (1909) in form of the differential equation

$$\frac{d\hat{z}}{dx} = b(a - \hat{z}) \qquad (1)$$

where a is the asymptotic value of z, b the factor of proportionality. As usual in statistics, $\hat{z}$ is the hypothetical value, z the experimental value of the crop-yield. For equation (1) it is assumed: No over-fertilization. For the case of overfertilization see Schneeberger (2009b).
Solution of formula (1) with boundary condition $\hat{z}(x = 0) = c$ is Mitscherlich's curve in the especial instructive form

$$\hat{z}(x) = c + (a - c)(1 - e^{-bx}) \qquad (2a)$$

demonstrated in figure 1 with c=0.5, a=1.2, b=0,75. For estimation of the parameters in practice see Schneeberger (2009a).

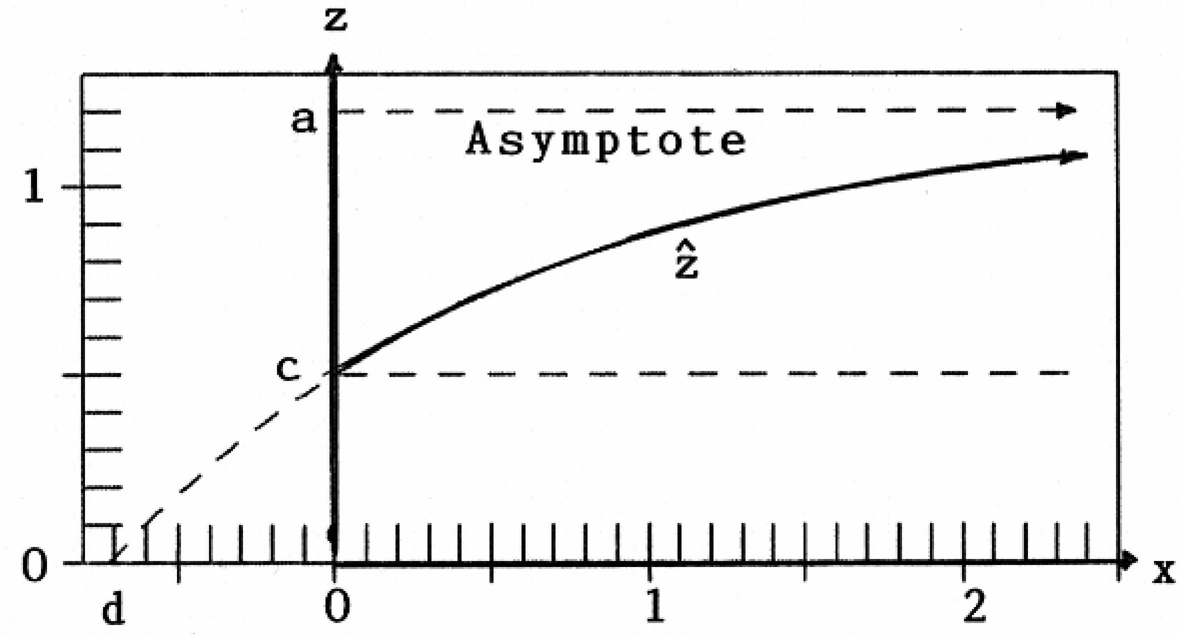

Figure 1: Crop-yield $\hat{z}(x)$ as function of one fertilizer x

An other form of formula (2a) is: $\hat{z}(x) = a - (a - c)e^{-bx}$ (2b)
used in the following.

Baule (1918) gave the solution of equation (1) in the form

$$\hat{z} = a(1 - e^{-b(x-d)}) \tag{3a}$$

with
$$d = -\frac{1}{b}\ln\frac{a}{a-c} \quad (<0) \tag{3b}$$

in figure 1 we have d=-0.719.

Generalization

Now we assume two variables (=fertilizers) x and y (see figure 2)

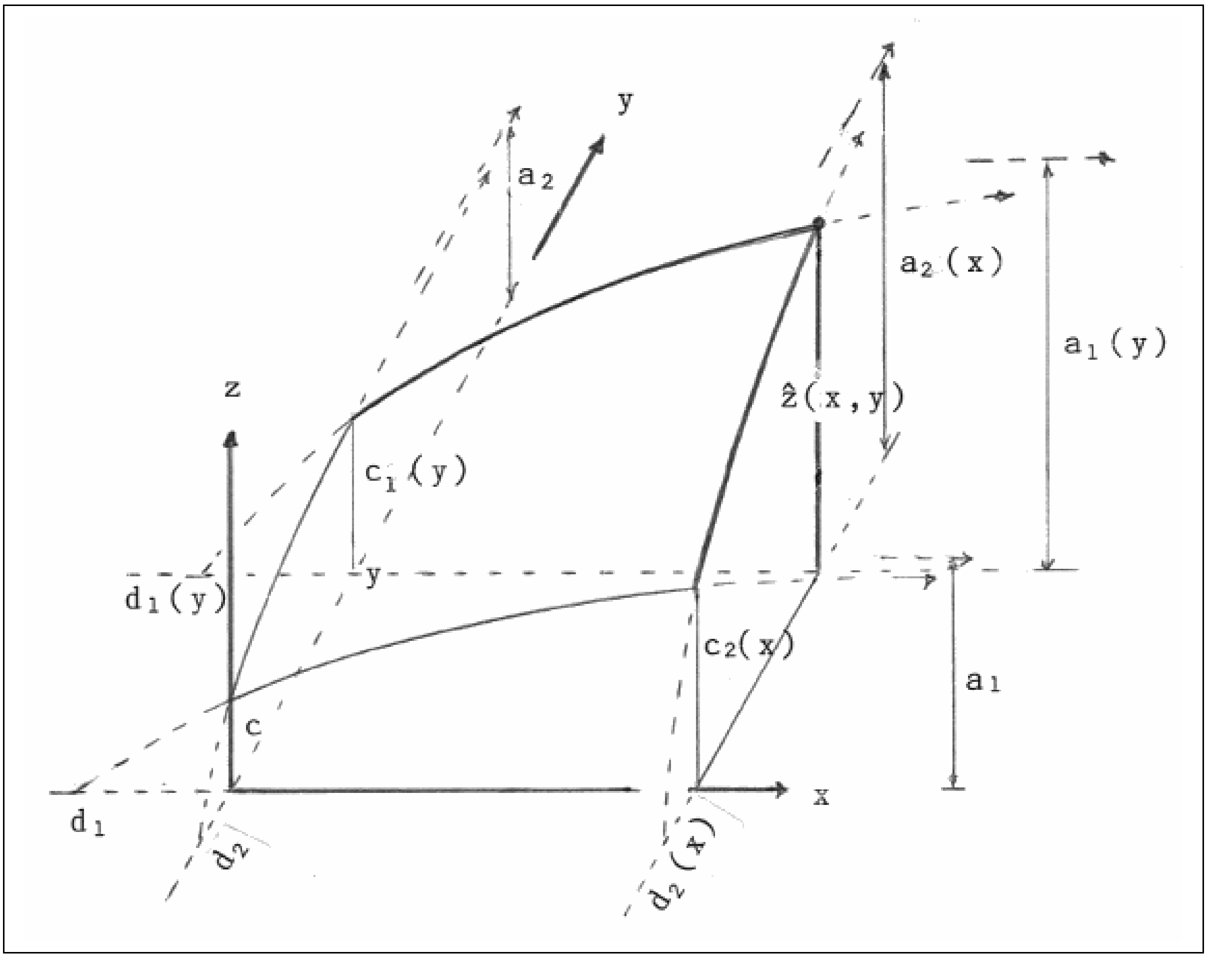

Figure 2: Crop-yield $\hat{z}(x, y)$ as function of two fertilizers x and y

Then we have for the crop-yield $\hat{z}(x,y)$ with formula (2b)

$$\hat{z}(0,y) = c_1(y) = a_2 - (a_2 - c)e^{-b_2 y} \qquad (4a)$$

and

$$\hat{z}(x,0) = c_2(x) = a_1 - (a_1 - c)e^{-b_1 x} \qquad (4b)$$

Note: For short we write $a_1(y=0) = a_1$, $a_2(x=0) = a_2$, $c_1(y=0) = c_2(x=0) = c$,
$b_1(y=0) = b_1$, $b_2(x=0) = b_2$,
$d_1(y=0) = d_1$, $d_2(x=0) = d_2$.
Generalizing formulae (4) we have

$$\hat{z}(x,y) = a_2(x) - (a_2(x) - c_2(x))e^{-b_2(x)y} \qquad (5a)$$

$$\hat{z}(x,y) = a_1(y) - (a_1(y) - c_1(y))e^{-b_1(y)x} \qquad (5b)$$

and herewith

$$\frac{a_2(x)}{a_1(y)} = \frac{1 - \dfrac{a_1(y) - c_1(y)}{a_1(y)} e^{-b_1(y)x}}{1 - \dfrac{a_2(x) - c_2(x)}{a_2(x)} e^{-b_2(x)y}} \qquad (6)$$

Now I make use of a result of Mitscherlich (1947): …“given different fertilizers, the parameter c (in Mitscherlich’s notation) is constant for a fixed fertilizer, as I could prove in tens of years of work”…. Mitscherlich’s “Wirkungsgrad” c is our parameter b except for the constant ln10. This means:

$$b_1(y) = b_1 \text{ and } b_2(x) = b_2 \text{ independent of } y \text{ resp. } x \qquad (7)$$

Herewith we get from formula (6):

$$\frac{a_1(y) - c_1(y)}{a_1(y)} = 1 - \frac{c_1(y)}{a_1(y)} \text{ is independent of y, i.e. } \frac{a_1(y)}{c_1(y)} = \frac{a_1(y=0)}{c_1(y=0)} = \frac{a_1}{c} \qquad (8a)$$

and

$$\frac{a_2(x)}{c_2(x)} = \frac{a_2}{c} \text{ independent of x} \qquad (8b)$$

and with this

$$d_1(y) = \frac{1}{b_1} \ln \frac{a_1(y)}{a_1(y) - c_1(y)} = \frac{1}{b_1} \ln \frac{a_1}{a_1 - c} = d_1 \text{ independent of y} \qquad (9a)$$

and

$$d_2(x) = d_2 \text{ independent of x.} \qquad (9b)$$

Finally we have from formula (5a) (or (5b)):

$$\hat{z}(x,y) = \frac{a_2}{c} c_2(x) - \frac{a_2 - c}{c} c_2(x) e^{-b_2 y} = \frac{1}{c} c_2(x)(a_2 - (a_2 - c)e^{-b_2 y}) = \frac{1}{c} c_2(x) c_1(y) \qquad (10)$$

or in the most instructive form

$$\hat{z}(x,y) = \frac{1}{c}(c + (a_1 - c)(1 - e^{-b_1 x}))(c + (a_2 - c)(1 - e^{-b_2 y})) \qquad (11)$$

RESULT: The generalized Mitscherlich formula in two variables is the product of the one-dimensional formulae, multiplied by 1/c.

With formulae (9) one can show, that formula (11) is identical with the formula of Baule (1918).

$$\hat{z}(x,y) = A(1-e^{-b_1(x-d_1)})(1-e^{-b_2(y-d_2)}) \quad \text{with} \quad A = \frac{a_1 a_2}{c} \tag{12}$$

Equations (10) and (11) can easily be generalized for n fertilizers:

$$\hat{z}(x_1,...x_n) = \frac{1}{c^{n-1}} \hat{z}(x_1,0,...0)\hat{z}(0,x_2,...0).....\hat{z}(0,0,...x_n) \tag{13}$$

Application

The following data are from an example of Steinhauser, Langbehn and Peters (1992) with x (in 100 kg/ha of P_2O_5), y (in 100 kg/ha of K_2O), z (in 1000kg/ha of rye).

Table 1: Crop-yield $\hat{z}(x,y)$ in 1000 kg/ha of rye, x in 100 kg/ha of P_2O_5, y in 100 kg/ha of K_2O

	$x_1 = 0.25$	$x_2 = 0.50$	$x_3 = 0.75$	$x_4 = 1.00$	$x_5 = 1.25$	$x_6 = 1.50$
$y_1 = 0.25$	1.00	1.22	1.41	1.58	1.73	1.87
$y_2 = 0.50$	1.41	1.79	2.09	2.34	2.55	2.73
$y_3 = 0.75$	1.71	2.21	2.59	2.90	3.15	3.35
$y_4 = 1.00$	2.00	2.55	2.98	3.32	3.59	3.81
$y_5 = 1.25$	2.22	2.82	3.29	3.65	3.94	4.18
$y_6 = 1.50$	2.41	3.05	3.55	3.93	4.24	4.50

The 5 parameters c, a_1, b_1, a_2, b_2 were determined with the method of Least Squares of Gauss

$$f(c,a_1,b_1,a_2,b_2) = \sum_x \sum_y (z(x,y) - \hat{z}(x,y))^2 \rightarrow \text{Min} \tag{14}$$

summing over all 36 data-points. The minimum was gained iteratively with the non-linear Simplex-Method of Nelder and Mead (1965). The result is

$$c=0.2715, \quad a_1=0.9472, \quad b_1 = 0.899, \quad a_2=1.9438, \quad b_2 = 1.027 \qquad \text{and}$$

$$\hat{z} = 3.6832(0.2715 + 0.6757(1-e^{-0.899x}))(0.2715 + 1.6723(1-e^{-1.027y}))$$

In figure 3a the contour-lines $\hat{z}(x,y)$=0.2, 0.4, …4.0 are drawn, in figure 3b intersecting curves $\hat{z}(x,y = \text{const.})$ for y=d_2,0, 0.5, 1.0 ,1.5 and ∞ are plotted. It is obvious that they are Mitscherlich-curves. We have $d_1 = -0.376$, d_2=--0.146. The asymptotes of the curves $\hat{z}(x,y = c)$ are horizontal dotted straight lines in figure 3b. Especially for y=∞ we get

$\hat{z}(\infty,\infty) = \frac{a_1 a_2}{c} = 6.78$, Baule's parameter A in formula (12). $\hat{z}(x, y=\infty)$ is curve $a_2(x)$ of figure 2.
In analogy intersecting curves for x=const. could be plotted.

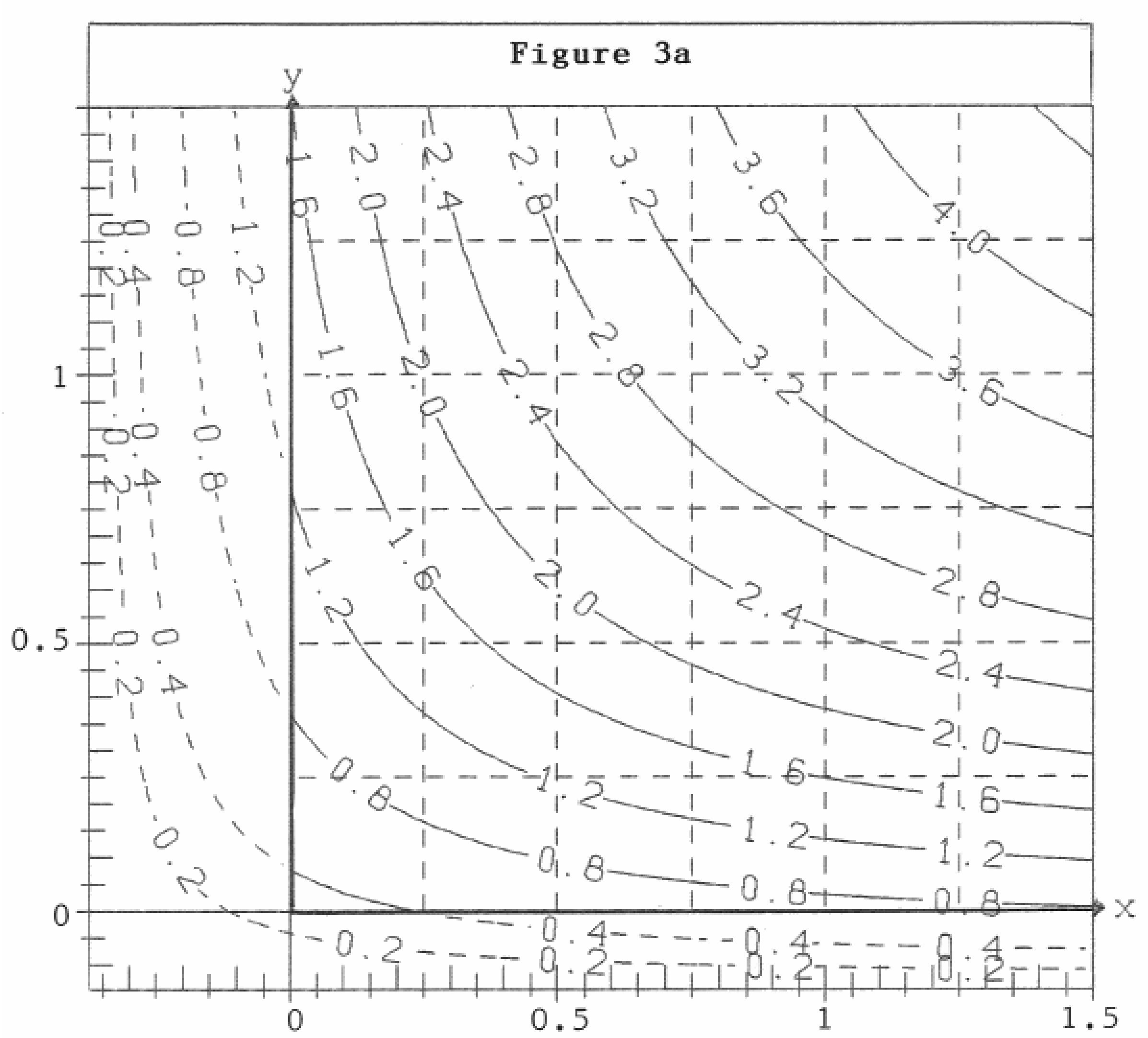

Figure 3a: Contour-lines $\hat{z}(x, y)$=const. - in 1000 kg/ha of rye, x in 100 kg/ha of P_2O_5, y in 100 kg/ha of K_2O

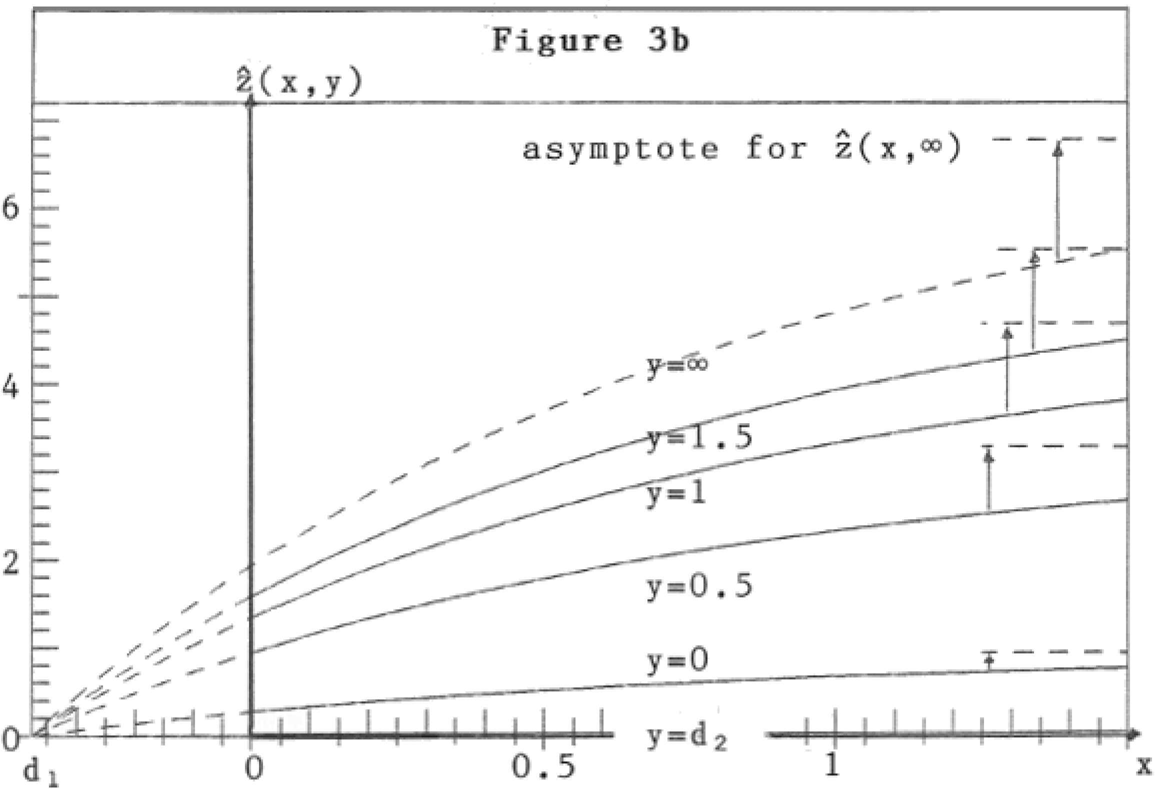

Figure 3b: Mitscherlich-curves $\hat{z}(x, y = \text{const.})$

Example: With fertilizer 100 kg/ha of P_2O_5 (x=1) and 50 kg/ha of K_2O (y=0.5) we get the crop-yield 2335 kg/ha of rye ($\hat{z} = 2.335$).. With x=y=0 (i.e. without external fertilizers) we would get 272 kg/ha of rye ($\hat{z} = c = 0.272$).

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A generalization with overfertilization is given in paper 5 (Paper 5: Mitscherlich's Law: Generalization with several Fertilizers and Overfertilization)

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