Mitscherlich's Law. A Supplement

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It was shown in www.soil-statistic.de, Paper 1 (Mitscherlich's Law: Sum of two exponential Processes. Conclusions), that Mitscherlich's curve: crop $\hat{y}$ in dependence on fertilizer x, can be partitioned into two exponential processes $\hat{y}_{01}$ and $\hat{y}_{02}$. It will be shown that this partition is indeed of special importance, but it is not the only possible one. There is one further partition of special importance, called here $\hat{y}_{11}$ and $\hat{y}_{12}$, and with these two special partitions an infinite number of others is given.
The "experiment", seed and crop, with certain soil, seed, fertilizer etc., as described in Paper 1, in general doesn't result in partition $\hat{y}_{01}$ and $\hat{y}_{02}$ (this must be corrected), but in one of these infinite partitions. If this one would be known, the loss of soil-immanent fertilizer by the crop and the degree of exploitation of the external fertilizer could be calculated.

Partitions of Mitscherlich's Formula

In Paper 1 already one particular partition of $\hat{y} = \hat{y}_{01} + \hat{y}_{02}$ with

$$\hat{y}_{01} = a(1 - e^{-bx}) \tag{9}$$
$$\hat{y}_{02} = ce^{-bx} \tag{10}$$

is given, see figure 1 and figure 3. $\hat{y}_{01}$ is the crop from the external fertilizer, $\hat{y}_{02}$ from the soil-immanent fertilizer. $\hat{y}_{01}$ is an exponential growing process with asymptote $\hat{y} = a$, $\hat{y}_{02}$ an exponential declining process with asymptote $\hat{y} = 0$. So $\hat{y}_{01}$ makes most (crop) of the external fertilizer. $\hat{y}_{02}$ makes least use of the soil-immanent fertilizer; (9) and (10) is the optimal partition..
The contrary is the case with the partition $\hat{y} = \hat{y}_{11} + \hat{y}_{12}$ with

$$\hat{y}_{11} = (a - c)(1 - e^{-bx}) \tag{12a}$$
$$\hat{y}_{12} = c \tag{12b}$$

see figure 3. Now the crop from the external fertilizer $\hat{y}_{11}$ has the asymptote $\hat{y} = a - c$, much worse than with $\hat{y}_{01}$; $\hat{y}_{12} = c$ means, that a maximum of the soil-immanent fertilizer is spent, independent of the quantity of the external fertilizer x. (12a/b) is the poorest partition.

But with partitions (9), (10) and (12a), (12b) we have an infinite number of further partitions $\hat{y} = \hat{y}_{\bullet 1} + \hat{y}_{\bullet 2}$ ($\bullet$ for 2, 3,…):

$$\hat{y}_{\bullet 1} = \alpha \hat{y}_{01} + (1 - \alpha)\hat{y}_{11} \tag{13a}$$
$$\hat{y}_{\bullet 2} = \alpha \hat{y}_{02} + (1 - \alpha)\hat{y}_{12} \tag{13b}$$
$$(0 \le \alpha \le 1)$$

$\alpha = 0$ gives the poorest, $\alpha = 1$ the optimal partition. I will call α parameter of affinity. Because of $\hat{y}_{\cdot 1} = \hat{y}_{11} + \alpha(\hat{y}_{01} - \hat{y}_{11})$ for formula (13a), $\hat{y}_{\cdot 1}(x)$ for fixed x grows from $\hat{y}_{11}(x)$ to $\hat{y}_{01}(x)$ with parameter α in a linear way. The same is right with $\hat{y}_{\cdot 2}$. See the α -scala in figure 3. In addition the curves $\hat{y}_{\cdot 1} = \hat{y}_{21}$ and $\hat{y}_{\cdot 2} = \hat{y}_{22}$ for $\alpha = 0.7$ are plotted there.

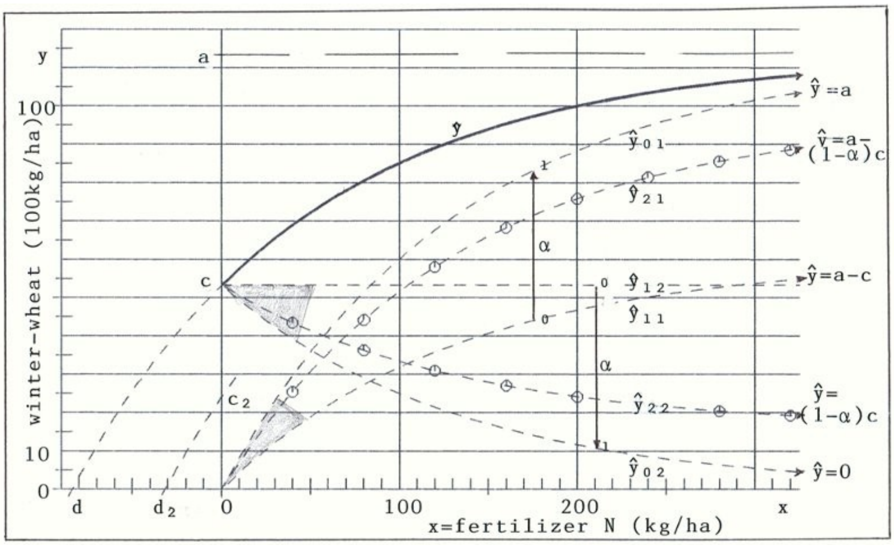

Figure 3: Partition of Mitscherlich's curve $\hat{y}$ into two components $\hat{y}_{i1}$ and $\hat{y}_{i2}$ (i=0,1,2,…)

For a special "experiment" a certain value of α will exist. The knowledge of this α would be of great importance for the knowledge of the loss of soil-immanent fertilizer and the effectiveness of the external fertilizer x. The greater α, the better for both results.
If α would be known (e.g. $\alpha = 0.7$), we would have for given x (e.g. 200 kg/ha of N): $\hat{y}_{22}(x) = c_2 = 24.13(100kg/ha)$ of winter-wheat, and herewith d_2 according to formula (11a) of paper 1: $d_2 = 31.65$ (kg/ha) of N is the soil-immanent fertilizer, needed for the total crop $\hat{y}$(x=200)=100(100kg/ha) of winter-wheat. In table 2 results for some further values of α are given.

Table 2: Relation between α and d_2 (x=200)

α	c_2	d_2
0.00	53.20	-83.80
0.60	28.28	-37.95
0.65	26.20	-34.76
0.70	24.13	-31.65
0.75	22.05	-28.60
0.80	19.98	-25.63
1	11.67	-14.36

In reversed direction we get α from d_2. So the problem of finding α is that of determining the value of d_2.

continued 09.01.2012: To demonstrate the dependence of $\hat{y}_1$ and $\hat{y}_2$ (in short for $\hat{y}_{\bullet 1}$ and $\hat{y}_{\bullet 2}$) on the parameter α, figures 4a, 4b… 4e give the curves $\hat{y}_1$, $\hat{y}_2$ and $\hat{y} = \hat{y}_1 + \hat{y}_2$ for $\alpha = 0, 0.25, 0.50, 0.75, 1$.

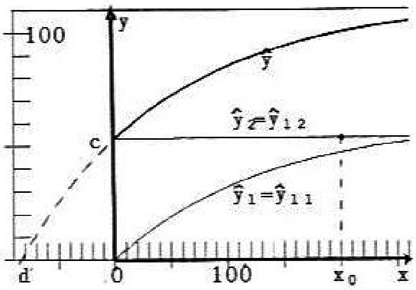

Fig. 4a: $\alpha = 0$; $c_2 = 53.2$, $d_2 = -83.8$

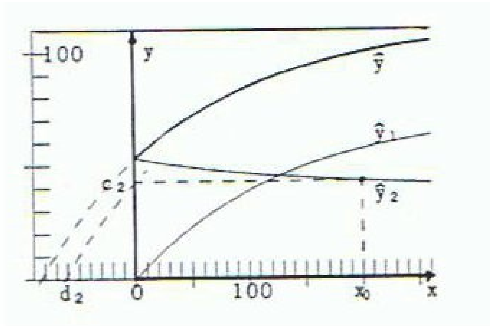

Fig. 4b: $\alpha = 0{,}25$; $c_2 = 42.8$, $d_2 = -62.7$

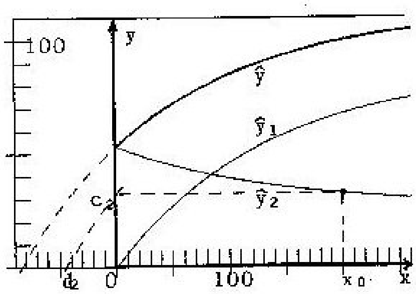

Fig. 4c: $\alpha = 0.5$; $c_2 = 32.44$, $d_2 = -44.6$

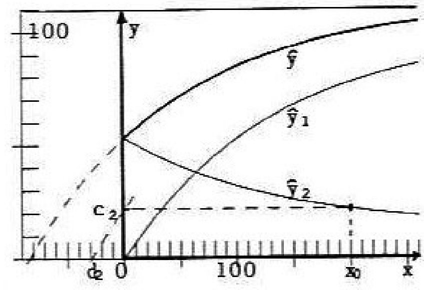

Fig. 4d: $\alpha = 0.75$; $c_2 = 22.1$, $d_2 = -28.6$

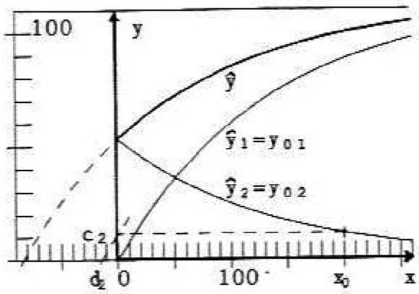

Fig. 4e: $\alpha = 1$; $c_2 = 11.7$, $d_2 = -14.4$

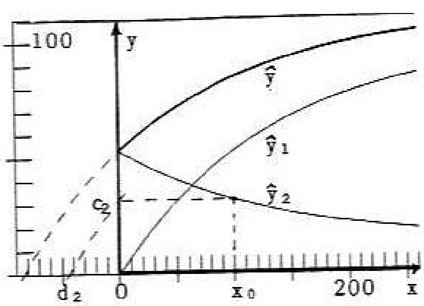

Fig. 5: $\alpha = 0.75$; $c_2 = 32.0$, $d_2 = -43.8$

The aim of external fertilization and soil-care must be maximizing $\hat{y}_1$ and minimising $\hat{y}_2$, or in short, maximizing α .
How can the value α of a fertilizer-soil combination be computed?

With the original data, given in paper 1, this cannot be done. For that the registration of (at least) one pair of data $(x_0, d_2(x_0))$, for example for x_0=200, is necessary; $-d_2(x_0) > 0$ is the quantity of soil-immanent fertilizer, which gives the part of crop $\hat{y}_2(x_0) = c_2(x_0)$.
$\Delta = -d - (-d_2(x_0)) > 0$ is the soil-immanent fertilizer after crop, which can be measured for example with a chemical analysis - as I assume. Herewith we get $d_2(x_0) = d + \Delta$. Then with $d_2(x_0)$ the value of $c_2(x_0)$ - signed in the figures as c_2 - is found as solution of

$$c_2 + (a - c_2)(1 - e^{-bd_2}) = 0, \quad \text{or} \quad c_2 = a(1 - e^{bd_2}) = \hat{y}_2(x_0) \tag{14}$$

and herewith (see figure 3)

$$\alpha = \frac{c - c_2}{c - ce^{-bx_0}} = \frac{c - \hat{y}_2(x_0)}{c - \hat{y}_{02}(x_0)} \tag{15}$$

If for example, an experiment with the above soil-fertilizer combination for external fertilizing with $x_0 = 200$ gives the value $d_2 = -62.7$, then the affinity is $\alpha = 0.25$ (see Fig.4b). $d_2 = -28.6$ gives $\alpha = 0.75$ (Fig.4d). One can see from figures 4b and 4d, that

1. The soil-fertilizer combination with higher value of α yields by far the better exploitation of the external fertilizer (see curves $\hat{y}_1$): for $x_0 = 200$ there is $\hat{y}_1(\alpha = 0.75)/\hat{y}_1(0.25) =$ 77.89/57.13=1.36; that means, that with the same quantity of external fertilizer $x_0 = 200$ the exploitation is 36 % higher.

2. To yield the same total crop $\hat{y}$, the part of crop $\hat{y}_2$ from the soil-immanent fertilizer reduces to about one half in our example: for x_0=200 we get $\hat{y}_2(\alpha = 0.75)/\hat{y}_2(\alpha = 0.25)$ $= c_2(\alpha = 0.75)/c_2(\alpha = 0.25) = 22.1/42.2 = 0.52$. This means: Only 45.6% of the soil-immanent fertilizer $(d_2(\alpha = 0.75)/d_2(\alpha = 0.25) = 28.6/62.7 = 0.456)$ is spent in a soil-fertilizer combination with $\alpha = 0.75$ against one with $\alpha = 0.25$. The quantity of external fertilizer is the same in both cases: x_0=200.

I think, this is active soil-conservation.

With n test points x_i(i=1,…,n) instead of the one x_0 we get values α_i, which are all estimates of the same “true” α (cf. figures 4d and 5 (with x_0=100)). So our final estimate of α then is $\sum \alpha_i / n$. For gaining the n experimental values α_i of course the assumptions of physical experiments must be fulfilled: All experimental parameters are constant, only the value of x is varied. This will be hard to realise in agronomy.

Further interesting questions would arise by changing another parameter, for example the soil, etc.

Acknowledgement
I praise the internet! It gives the possibility to publish works, which are out of the ordinary. I choose this way, after my first paper was returned with the comment "is in its contents fully out of the scope of the journal" (translated from German).